Effective Gap Equation for the Inhomogeneous LOFF Superconductive Phase

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We present an approximate gap equation for different crystalline structures of the LOFF phase of high density QCD at T=0. This equation is derived by using an effective condensate term obtained by averaging the inhomogeneous condensate over distances of the order of the crystal lattice size. The approximation is expected to work better far off any second order phase transition. As a function of the difference of the chemical potentials of the up and down quarks, $\delta\mu$, we get that the octahedron is energetically favored from $\delta\mu = \Delta_0/\sqrt{2}$ to $0.95\Delta_0$, where Δ_0 is the gap for the homogeneous phase, while in the range $0.95\Delta_0 - 1.32\Delta_0$ the face centered cube prevails. At $\delta\mu = 1.32\Delta_0$ a first order phase transition to the normal phase occurs.

I. INTRODUCTION

The existence of an inhomogeneous superconductive phase characterized by a periodic structure of the gap parameter was hypothesized long ago in two separate and contemporary papers by Larkin and Ovchinnikov [1] and by Fulde and Ferrel [2] and is therefore named LOFF phase. These original papers studied superconductors in presence of a strong magnetic field coupled to the spins of the conduction electrons. The corresponding interaction hamiltonian gives rise to a separation of the Fermi surfaces of the pairing electrons since they have opposite spins. If the separation is very high the pairing is destroyed and there is a transition from the superconductive to the normal state. In [1] and [2] it was shown that a new state could be formed, close to the transition line. This state has the feature of exhibiting an order parameter, the gap, which is not a constant, but has a space variation whose typical wavelength is of the order of the inverse of the difference in the Fermi energies of the pairing electrons. The space modulation of the gap arises because the electron pair has non zero total momentum. In the simplest case the modulation is that of a plane wave. In the general case it is given by the sum of several plane waves, which might lead to a crystalline structure. In any case the LOFF phase breaks translational and rotational symmetries.

Inhomogeneous superconductivity in metals has been the object of intense experimental investigations especially in the last decade, though the evidence is still inconclusive (for a review see [3]). Quite recently it has been also realized that at moderate density the mass difference between the strange and the up and down quarks at the weak equilibrium leads to a difference in their Fermi momenta, which renders in principle the LOFF state possible in color superconductivity [4] (for general reviews of color superconductivity see [5, 6, 7, 8, 9, 10, 11]). The study of the crystalline color superconductive phase of QCD is theoretically important for the understanding of the QCD phase diagram. It may be also relevant to the astrophysics of compact stars. As a matter of fact whereas at extreme densities one expects the color-flavor-locking phase to occur, at lower densities, within the inner core of compact stars, there is a better chance that the crystalline phase be realized. In these conditions a difference in the chemical potentials of the up and down quark might be generated by beta-equilibrium of the quark matter. While difficult to be studied in the laboratory this phenomenon might be therefore relevant in astrophysics, for example by explaining the sudden variations (glitches) of the rotation period of the pulsars (for a review of color superconductivity in compact stars see e.g.[12], [13]). In [14] a Ginzburg-Landau (GL) analysis of several crystalline structures was performed for QCD at $T = 0$. This approximation holds for small values of the gap parameter and therefore this analysis studies a small region of the phase diagram, i.e. the corner where the difference of the chemical potentials of the pairing quarks $\delta\mu$ is near to $\delta\mu_2$, the value where a second order phase transition from the LOFF to the normal phase is assumed to exist. For the one-plane-wave structure $\delta\mu_2$ is not far from the Clogston-Chandrasekhar [15], [16] point $\delta\mu_1 = \Delta_0/\sqrt{2}$ (Δ_0 is the value of the gap for the superconductive homogeneous phase). This is approximately the point where a first order transition occurs between the homogeneous Bardeen-Cooper-Schrieffer (BCS) and the inhomogeneous LOFF phases. In other cases $\delta\mu_2$ can be larger and the phase transition can be first-order.

An effective field theory for the crystalline color superconductive phase of QCD was developed in [17]. The papers [18] and [19] studied the properties of the Nambu-Goldstone Bosons (NGB) associated to the breaking of the translation symmetry, whose properties were analyzed in [20], where an approximation scheme valid at the GL point ($\Delta \to 0$) was developed. In the present paper we come back to this problem with the aim to implement an approximation scheme

for the gap equation along similar lines. We will find that an effective gap equation can be indeed written down, but it is expected to work better for large Δ, for example close to the BCS/LOFF transition point $\delta\mu_1$. We will work at $T = 0$ and for high densities, since we wish to study the LOFF phase in the context of QCD, where the physically relevant case is presumably the zero temperature high density inner core of compact stars. We will not consider a complete analysis, which is of course an endless task, but we shall limit our analysis to four crystalline structures. In Section II we discuss the one plane wave as the prototype case. Then we consider those cases that various analyses [1], [14], [21] indicate among the favored forms of the condensate: the two-antipodal plane wave, the octahedron and the face-centered-cube, to be discussed in Sections III and IV. The numerical results of Section IV are that, as a function of the difference $\delta\mu$ between the quark chemical potentials, the octahedron is energetically favored from $\delta\mu = \Delta_0/\sqrt{2}$ to $0.95\Delta_0$, while in the range $0.95\Delta_0 - 1.32\Delta_0$ the face centered cube prevails. After this point the normal phase is restored.

II. THE EFFECTIVE GAP EQUATION FOR THE FULDE-FERREL (ONE-PLANE WAVE) STATE

We begin our study by a review of one-plane wave, Fulde-Ferrel (FF) state. Although this case can be solved exactly, it is useful to consider it here in order to fix the notations and introduce some definitions to be used later on. We shall consider Cooper pairing of the massless quarks up u and down d, with chemical potential μ_u, μ_d. We define $\mu = (\mu_u + \mu_d)/2$ and $\delta\mu = |\mu_u - \mu_d|/2 \ll \mu$.

The gap equation, as discussed in Ref. [3], can be derived using the following effective interaction Lagrangian

$$\mathcal{L}_I = -\frac{3}{8}g\bar{\psi}\gamma^\mu\lambda^a\psi\,\bar{\psi}\gamma^\mu\lambda^a\psi\,. \tag{1}$$

Here g is a coupling constant, λ_a are color matrices and a sum over flavors is understood.

Following the same steps as in [3] we get that, in the mean field approximation, $\mathcal{L}_I$ in (1) is substituted by

$$\mathcal{L}_{cond} = -\frac{1}{2}\epsilon_{\alpha\beta3}\epsilon^{ij}(\psi_i^\alpha\psi_j^\beta\,\Delta(\boldsymbol{r}) + \text{c.c.}) + (L \to R) - \frac{1}{g}\Delta(\boldsymbol{r})\Delta^*(\boldsymbol{r})\,, \tag{2}$$

where i, j are flavor indices and α, β are color indices. In the FF state the total momentum of the Cooper pair is $2\mathbf{q}$ and the condensate has the space-dependence of a single plane wave [2]:

$$\Delta(\boldsymbol{r}) = \Delta e^{2i\mathbf{q}\cdot\mathbf{r}}\,. \tag{3}$$

It follows that at zero temperature, the gap equation for the FF state can be written as follows

$$1 = \frac{g\rho}{2}\int\frac{d\mathbf{v}}{4\pi}\int_0^\delta\frac{d\xi}{\sqrt{\xi^2 + \Delta^2}}\left(1 - \theta(-E_u) - \theta(-E_d)\right)\,, \tag{4}$$

where $|\mathbf{v}| = 1$ and δ is the ultraviolet cutoff that we assume equal to $\mu/2$; moreover we assume $\delta \gg q \sim \delta\mu$. The following equations

$$E_{u,d} = \pm\delta\mu \mp \mathbf{q}\cdot\mathbf{v} + \sqrt{\xi^2 + \Delta^2} \tag{5}$$

provide the quasi-particle dispersion laws and ρ is the density of states that in QCD with two flavor quarks is

$$\rho = \frac{4\mu^2}{\pi^2}\,. \tag{6}$$

We observe that

$$1 - \theta(-E_u) - \theta(-E_d) = \theta(E_u)\theta(E_d) \tag{7}$$

so that the integration in Eq.(4) is over the pairing region (PR), defined by

$$PR = \{(\xi, \mathbf{v})\,|\,E_u > 0\,\text{and}\,E_d > 0\}\,. \tag{8}$$

More explicitly the pairing region (PR) is defined by the condition $(\hat{\mathbf{q}} = \mathbf{q}/q)$

$$\text{Max}\left\{-1, z_q - \frac{\sqrt{\xi^2 + \Delta^2}}{q}\right\} < \mathbf{v}\cdot\hat{\mathbf{q}} < \text{Min}\left\{1, z_q + \frac{\sqrt{\xi^2 + \Delta^2}}{q}\right\} \tag{9}$$

with

$$z_q = \frac{\delta\mu}{q} \ . \tag{10}$$

Eq. (4) can be written in a way that will be useful later:

$$\Delta = i\frac{g\rho}{2} \int \frac{d\mathbf{v}}{4\pi} \int_0^\delta \frac{d\xi}{2\pi} \int d\ell_0 \ \frac{\Delta_{eff}}{\ell_0^2 - \xi^2 - \Delta_{eff}^2} \ , \tag{11}$$

that after energy integration gives

$$\Delta = \frac{g\rho}{2} \int \frac{d\,\mathbf{v}}{4\pi} \int_0^\delta d\xi \ \frac{\Delta_{eff}}{\sqrt{\xi^2 + \Delta_{eff}^2}} \ . \tag{12}$$

Here $\Delta_{eff} = \Delta_{eff}(\mathbf{v} \cdot \hat{\mathbf{q}}, \xi)$ is defined as

$$\Delta_{eff} = \Delta\theta(E_u)\theta(E_d) = \begin{cases} \Delta & \text{for } (\xi, \mathbf{v}) \in PR \\ \\ 0 & \text{elsewhere} \end{cases} \tag{13}$$

To the interaction term (2) in the lagrangian one should add the Lagrangian for free quarks. We adopt for its description the formalism of the High Density Effective Theory (HDET) (see [22, 23, 24, 25, 26], and, for reviews, [11, 27, 28]) and write it as follows

$$\mathcal{L}_0 = \sum_{\vec{v}} \left[\psi_+^\dagger iV \cdot \partial\psi_+ \ + \ \psi_-^\dagger i\tilde{V} \cdot \partial\psi_- \right] \ + \ (L \to R) \ . \tag{14}$$

Here the sum represents an average over velocities. The velocity dependent left-handed fields $\psi_\pm \equiv \psi_{\pm\mathbf{v}}$ are the positive energy part in the decomposition

$$\psi(x) = \int \frac{d\mathbf{v}}{4\pi} e^{-i\mu v \cdot x} \left[\psi_{\mathbf{v}}(x) + \Psi_{\mathbf{v}}(x) \right] \tag{15}$$

while $\Psi_{\mathbf{v}}$ is the negative energy part which has been integrated out. $\psi_{\mathbf{v}}$ is given by

$$\psi_{\mathbf{v}}(x) = e^{i\mu v \cdot x} P_+ \psi(x) = \int_{|\ell|<\delta} \frac{d^4\ell}{(2\pi)^4} e^{-i\ell \cdot x} P_+ \psi(\ell) \tag{16}$$

and therefore contains the residual momentum ℓ, corresponding to the decomposition of the quark momentum $p = \mu v + \ell$, with $v^\mu = (0, \mathbf{v})$ and $\ell_\parallel = \boldsymbol{\ell} \cdot \mathbf{v} = \xi$. We also introduce $V^\mu = (1, \mathbf{v})$ and $\tilde{V}^\mu = (1, -\mathbf{v})$. $P_\pm$ are projectors defined by

$$P_\pm = \frac{1}{2} \left(1 \pm \gamma_0 \boldsymbol{\gamma} \cdot \mathbf{v} \right) \ . \tag{17}$$

In this formalism we get from (2) ($C = i\sigma_2$ is the charge conjugation matrix)

$$\begin{aligned} \mathcal{L}_{cond} = \ & -\frac{\Delta}{2} \sum_{\mathbf{v_i}, \mathbf{v_j}} \exp\{i\mathbf{r} \cdot \boldsymbol{\alpha}(\mathbf{v_i}, \mathbf{v_j}, \mathbf{q})\} \epsilon_{\mathbf{ij}} \epsilon_{\alpha\beta 3} \psi_{\mathbf{v_i}; i\alpha}^{\mathbf{T}}(\mathbf{x}) \mathbf{C} \psi_{-\mathbf{v_j}; j\beta}(\mathbf{x}) \\ & -(L \to R) + \text{h.c.} - \frac{1}{g}\Delta(\boldsymbol{r})\Delta^*(\boldsymbol{r}) \ , \end{aligned} \tag{18}$$

where

$$\boldsymbol{\alpha}(\mathbf{v_i}, \mathbf{v_j}, \mathbf{q}) = 2\mathbf{q} - \mu_i\mathbf{v_i} - \mu_j\mathbf{v_j} \ . \tag{19}$$

This Lagrangian can be put, in momentum space, in the following form

$$\mathcal{L}_{int} = -\frac{1}{2} \sum_{\vec{v}} \Delta_{eff} \ \epsilon_{ij}\epsilon_{\alpha\beta 3} \psi_{\mathbf{v}; i\alpha}^T(\ell) C\psi_{-\mathbf{v}; j\beta}(-\ell) - (L \to R) + \text{h.c.} - \frac{1}{g}\Delta\Delta^* \ , \tag{20}$$

by an average procedure (see below) and then we can obtain the gap equation (11). It is useful to introduce the following basis for the fermion fields:

$$\psi_{+,\alpha i} = \sum_{A=0}^{3} \frac{(\sigma_A)_{\alpha i}}{\sqrt{2}} \psi_+^A, \ (i, \alpha = 1, 2), \quad \psi_{+,31} = \psi_+^4, \quad \psi_{+,32} = \psi_+^5 , \tag{21}$$

where σ_A are the Pauli matrices for $A = 1, 2, 3$ and $\sigma_0 = 1$ (similar expressions hold for ψ_-^A) and then we obtain that the propagator for the velocity dependent fermionic fields is given by

$$D_{AB}(\ell) = \frac{1}{V \cdot \ell \, \tilde{V} \cdot \ell - \Delta_{eff}^2} \begin{pmatrix} \tilde{V} \cdot \ell \, \delta_{AB} & -\Delta_{AB} \\ -\Delta_{AB} & V \cdot \ell \, \delta_{AB} \end{pmatrix} . \tag{22}$$

The matrix Δ_{AB} is as follows: $\Delta_{AB} = 0$ (A or $B = 4$ or 5), and, for $A, B = 0, ..., 3$:

$$\Delta_{AB} = \Delta_{eff} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{AB} . \tag{23}$$

To obtain the Lagrangian (20) one can perform a weighted average of the Lagrangian (18) over a region of the size of the lattice cell. The weight function $g_R(\mathbf{r})$ will be defined below. We note that in the gap equation the relevant momenta are small with respect to the gap which is of the order of q. Therefore we may assume that the velocity dependent fields are slowly varying over regions of the order of the lattice size. This means that in the average we can treat them as constant, and in conclusion the average is made only on the coefficient $\exp\{i\mathbf{r} \cdot \boldsymbol{\alpha}\}$. Therefore what we are computing is

$$I(\boldsymbol{\alpha}) = \Big\langle \exp\{i\mathbf{r} \cdot \boldsymbol{\alpha}\} g_R(\mathbf{r}) \Big\rangle \tag{24}$$

where the bracket means average over the cell.

It is possible to choose $g_R(\mathbf{r})$ in such a way that

$$I(\boldsymbol{\alpha}) = \delta_R^3 \left(\frac{\boldsymbol{\alpha}}{2q} \right) \tag{25}$$

where

$$\delta_R^3(\mathbf{x}) = \begin{cases} 1 & \text{for } |\mathbf{x}| < \dfrac{\pi}{2R} , \\ 0 & \text{elsewhere}. \end{cases} \tag{26}$$

An approximate expression of $g_R(\mathbf{r})$ is given by

$$g_R(\mathbf{r}) = \prod_{k=1}^{3} \frac{\sin\left[\dfrac{\pi q r_k}{R}\right]}{\pi r_k} . \tag{27}$$

For $R/\pi \approx 1$, g_R is different from zero only in a region of the size of the unit lattice cell; the resulting integral I obtained by (24) and (27) is reported (for the one-dimensional case and for $R = 2\pi/3$) in fig.1.

Independently of the exact form of $g_R(\mathbf{r})$, we will assume that the average procedure gives as a result the brick-shaped function δ_R defined in (26). Clearly this assumption is a crucial ingredient of our approximation.

In the $\mu \to \infty$ limit by taking $\mathbf{q}$ along the $z-$axis, we get for the components x and y of $\boldsymbol{\alpha}$

$$|(\mu_1 v_1 + \mu_2 v_2)_{x,y}| < \frac{\pi q}{R} , \tag{28}$$

i.e. approximately (for $\delta\mu \ll \mu$)

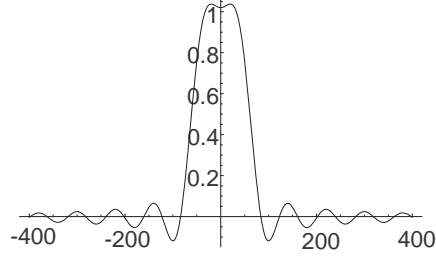$$|(v_1 + v_2)_{x,y}| < \frac{\pi q}{R\mu} . \tag{29}$$

FIG. 1: $I(x)$ as obtained by Equations (24) and (27) (in one dimension) for $q = 40$ MeV and $R = 2\pi/3$.

From this in the high density limit it follows

$$\mathbf{v_1} = -\mathbf{v_2} + \mathcal{O}(1/\mu)\,. \tag{30}$$

We use the x and y components of the δ_R function to get rid of the integration over $\mathbf{v_2}$ in eq. (18). The two factors π/R arising from the x and y components are absorbed into a wave function renormalization of the quark fields, both in the kinetic and in the gap terms. The z component gives a factor $\delta_R[h(\mathbf{v} \cdot \hat{\mathbf{q}})]$, where the function h was computed in [20]. Since here we work in the limit $\delta\mu \ll \mu$, we neglect corrections of order $\delta\mu/\mu$ and we use the asymptotic expression [20]:

$$h(\mathbf{v} \cdot \hat{\mathbf{q}}) = 1 - \frac{z_q}{\mathbf{v} \cdot \hat{\mathbf{q}}}\,. \tag{31}$$

After this average we get that the coefficient of the bilinear term in Eq.(19) is

$$\Delta\,\delta_R[h(\mathbf{v} \cdot \hat{\mathbf{q}})]\,. \tag{32}$$

This factor coincides with $\Delta_{eff}(\mathbf{v} \cdot \hat{\mathbf{q}}, \xi)$ if one chooses

$$R = \frac{\pi|\delta\mu - \mathbf{v} \cdot \mathbf{q}|}{2\sqrt{\xi^2 + \Delta^2}|h(\mathbf{v} \cdot \hat{\mathbf{q}})|}\,. \tag{33}$$

This is consistent with our hypothesis, $R/\pi \approx 1$, only if Δ is not too small, meaning that we should be far from a second order phase transition.

Let us finally notice that Eq. (12) can be obtained from Eq. (11) by substituting the previous expression for R with

$$R = \frac{\pi|\delta\mu - \mathbf{v} \cdot \mathbf{q}|}{2|\ell_0| \cdot |h(\mathbf{v} \cdot \hat{\mathbf{q}})|}\,. \tag{34}$$

In fact, observing that in any case Δ_{eff} is equal to 0 or Δ, at the pole we get back the expression (33). Then written as in (34), R and analogously Δ_{eff} become functions of the velocity and the energy; therefore our average should be better taken in the momentum space.

III. THE EFFECTIVE GAP EQUATION AT $T = 0$ FOR GENERIC CRYSTALLINE STRUCTURES

Let us now consider the general case of P plane waves

$$\Delta(\boldsymbol{r}) = \sum_{m=1}^{P} \Delta_m e^{2i\mathbf{q_m} \cdot \mathbf{r}}\,. \tag{35}$$

We shall consider only the case $\Delta_m = \Delta$ for any $m = 1, ...P$ and

$$\mathbf{q_m} = q\,\mathbf{n_m}\,, \tag{36}$$

with $\mathbf{n_m}$ unit vectors. Generalizing the results of the previous equations we substitute in the Lagrangian (20) and in the propagator (22) $\Delta_{eff}(\mathbf{v} \cdot \mathbf{n}, \ell_0)$ with

$$\Delta_E(\mathbf{v}, \ell_0) = \sum_{m=1}^{P} \Delta_{eff}(\mathbf{v} \cdot \mathbf{n_m}, \ell_0) \,. \tag{37}$$

In this equation we have made explicit the dependence on the energy ℓ_0 arising from the average. Correspondingly the gap equation is

$$P\Delta = i\frac{g\rho}{2} \int \frac{d\mathbf{v}}{4\pi} \int \frac{d^2\ell}{2\pi} \frac{\Delta_E(\mathbf{v}, \ell_0)}{\ell_0^2 - \ell_{\parallel}^2 - \Delta_E^2(\mathbf{v}, \ell_0)} \tag{38}$$

which generalizes Eq. (11). The origin of the factor P on the l.h.s. of this equation is as follows. The Lagrangian contains the term

$$\frac{\Delta^*(\mathbf{r})\Delta(\mathbf{r})}{g} \tag{39}$$

which, when averaged over the cell, gets non vanishing contribution only from the diagonal terms in the double sum over the plane waves.

The energy integration is performed by the residue theorem and the phase space is divided into different regions according to the pole positions, defined by

$$\epsilon = \sqrt{\xi^2 + \Delta_E^2(\mathbf{v}, \epsilon)} \,. \tag{40}$$

Therefore we get

$$P\Delta \ln \frac{2\delta}{\Delta_0} = \sum_{k=1}^{P} \int \int_{P_k} \frac{d\mathbf{v}}{4\pi} d\xi \frac{\Delta_E(\mathbf{v}, \epsilon)}{\sqrt{\xi^2 + \Delta_E^2(\mathbf{v}, \epsilon)}} = \sum_{k=1}^{P} \int \int_{P_k} \frac{d\mathbf{v}}{4\pi} d\xi \frac{k\Delta}{\sqrt{\xi^2 + k^2\Delta^2}} \tag{41}$$

where the regions P_k are defined as follows

$$P_k = \{(\mathbf{v}, \xi) \,|\, \Delta_E(\mathbf{v}, \epsilon) = k\Delta\} \tag{42}$$

and we have made use of the equation

$$\frac{2}{g\rho} = \ln \frac{2\delta}{\Delta_0} \tag{43}$$

relating the BCS gap Δ_0 to the four fermion coupling g and the density of states. The first term in the sum, corresponding to the region P_1, has P equal contributions with a dispersion rule equal to the Fulde and Ferrel case. This can be interpreted as a contribution from P non interacting plane waves. In the other regions the different plane waves have an overlap. Since the definition of the regions P_k depends on the value of Δ, their determination is part of the problem of solving the gap equation.

Let us comment on our result. We have shown that in our approximation the dispersion relation for the quasi-particles has several branches corresponding to the values $k\Delta$, $k = 1, \cdots, P$. Therefore the following interpretation of the gap equation (41) can be given. Each term in the sum corresponds to one branch of the dispersion law, i.e. to the propagation of a gapped quasi-particle with gap $k\Delta$. The corresponding region is nothing but P_k. However, the regions P_k do not represent a partition of the phase space since it is possible to have at the same point quasi-particles with different gaps.

As a final remark, it is possible to test our method in an exactly solvable case, i.e. the particular case of P identical plane waves. In this case the only non empty region in the gap equation is P_P therefore the gap equation reduces to the FF gap equation for a gap $P\Delta$, i.e. the expected result. We assume this as a consistency test of our approximation.

IV. NUMERICAL ANALYSIS

The free energy Ω is obtained by integrating in Δ the gap equation. At fixed $\delta\mu$, Ω is a function of Δ and z_q, therefore the energetically favored state satisfies the conditions

$$\frac{\partial\Omega}{\partial\Delta} = 0 \,, \qquad \frac{\partial\Omega}{\partial z_q} = 0 \,, \tag{44}$$

and must be the absolute minimum. We now analyze three different structures by this criterion. We will assume $\mu = 400$ MeV and $\delta = \mu/2$.

A. Strip

This is the case of two antipodal plane waves, i.e. we take $P = 2$ in Eq.(35) and

$$\mathbf{n}_1 = (0, 0, +1), \qquad \mathbf{n}_2 = (0, 0, -1), \tag{45}$$

in Eq.(36). If ϵ is the pole position in the ℓ_0 complex plane, the gap equation is

$$2\Delta \ln \frac{2\delta}{\Delta_0} = \int \int_{P_1} \frac{d\mathbf{v}}{4\pi} d\xi \frac{\Delta_E(\mathbf{v}, \epsilon)}{\sqrt{\xi^2 + \Delta_E^2(\mathbf{v}, \epsilon)}} + \int \int_{P_2} \frac{d\mathbf{v}}{4\pi} d\xi \frac{\Delta_E(\mathbf{v}, \epsilon)}{\sqrt{\xi^2 + \Delta_E^2(\mathbf{v}, \epsilon)}}. \tag{46}$$

Here $\displaystyle\int_{P_1}$ represents a region where the two Δ_{eff} appearing in Δ_E have no overlap. In this region

$$\Delta_E(\mathbf{v}, \epsilon) = \Delta \left\{ \theta(E_u^1)\theta(E_d^1) + \theta(E_u^2)\theta(E_d^2) - 2\theta(E_u^1)\theta(E_d^1)\theta(E_u^2)\theta(E_d^2) \right\}_\Delta = \Delta. \tag{47}$$

In other words

$$\int \int_{P_1} \frac{d\mathbf{v}}{4\pi} d\xi \frac{\Delta_E(\mathbf{v}, \epsilon)}{\sqrt{\xi^2 + \Delta_E^2(\mathbf{v}, \epsilon)}} = \Delta \int \int \frac{d\mathbf{v}}{4\pi} d\xi \frac{\left\{ \theta(E_u^1)\theta(E_d^1) + \theta(E_u^2)\theta(E_d^2) - 2\theta(E_u^1)\theta(E_d^1)\theta(E_u^2)\theta(E_d^2) \right\}_\Delta}{\sqrt{\xi^2 + \Delta^2}}. \tag{48}$$

On the other hand the integral over P_2 represents the region where the two Δ_{eff} appearing in Δ_E do overlap. In this region $\Delta_E^2(\mathbf{v}, \epsilon) = 4\Delta^2$ and we have

$$\int \int_{P_2} \frac{d\mathbf{v}}{4\pi} d\xi \frac{\Delta_E(\mathbf{v}, \epsilon)}{\sqrt{\xi^2 + \Delta_E^2(\mathbf{v}, \epsilon)}} = \Delta \int \int \frac{d\mathbf{v}}{4\pi} d\xi \frac{2 \left\{ \theta(E_u^1)\theta(E_d^1)\theta(E_u^2)\theta(E_d^2) \right\}_{2\Delta}}{\sqrt{\xi^2 + 4\Delta^2}}, \tag{49}$$

where the subscript 2Δ on the r.h.s. means that in the dispersion laws $E_{u,d}^{1,2}$ one has to use Eqns. (5) with $\Delta \to 2\Delta$. As already observed, a test of our approximation is given by the fact that the gap equation obtained in this way reproduces in the case of $\mathbf{q}_1 = \mathbf{q}_2$ the FF result.

As we observed above, we expect more reliable results for large Δ, i.e. far off the GL point. Nevertheless we can determine the phase transition point $\delta\mu_2$ within this approximation; we find $\delta\mu_2 \approx 0.83\Delta_0$, which compares fairly well with the exact result in the weak coupling limit, i.e. $0.75\Delta_0$; in our approximation we find a first order phase transition (see subsection IV C for further comments).

It may be useful to have a pictorial representation of the various regions. They are reported for the strip in a $(z = \cos\theta, \xi)$ plane in Fig. 2, at $\delta\mu = \delta\mu_1$, where our approximation is expected to work better.
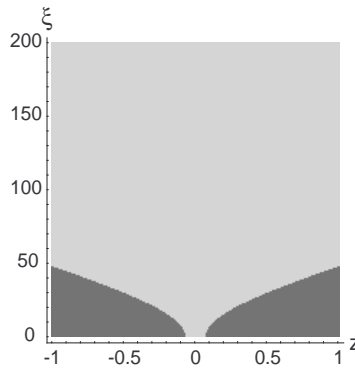

FIG. 2: The regions P_1 and P_2 for the strip in the (z, ξ) plane with $z = \cos\theta = \mathbf{v} \cdot \hat{\mathbf{q}}_1$. The region P_1 is the dark region, whereas the region P_2 covers all the plane and is in light grey. Here $\Delta = 0.75\Delta_0$, $z_q = 1.0$, $\delta\mu = \delta\mu_1$; ξ in MeV.

To get a better insight of the main features of our approximation, we can compare the approximate dispersion law we are using with the exact one. The exact dispersion law for the strip was derived in [1] (see also [29]; in both papers the energy is computed for each species from the corresponding Fermi surface):

$$E^1_{u,d} = \pm\delta\mu \mp \mathbf{q}\cdot\mathbf{v} + w|\xi|, \quad E^2_{u,d} = \pm\delta\mu \pm \mathbf{q}\cdot\mathbf{v} + w|\xi| \tag{50}$$

where

$$w^{-1} = I_0\left(\frac{2\Delta}{q\cos\theta}\right) . \tag{51}$$

I_0 is the Bessel function and θ the angle between the $z-$axis and $\mathbf{q} = \mathbf{q_1}$.

This comparison for the strip is reported in Fig. 3 (we use the parameters $\Delta = 0.75\Delta_0$, $\delta\mu = \delta\mu_1$ and $z_q \approx 0.8$, which is more appropriate for comparison with the exact result). On the right diagram the gray region depicts the area where coupling is more favorable, because here both quasiparticles with gap Δ and 2Δ contribute; to this area corresponds on the left diagram, in gray, the pairing region, where $E^k_{u,d} > 0$ ($k = 1,2$). The white areas depict respectively, on the right an area where coupling is less favorable, because only quasiparticles with gap 2Δ can contribute, and on the left the blocking region, where the energies are negative. We have limited the analysis to the region of smaller ξ since this is the region more relevant for coupling. The two graphs do not coincide, which is due to the approximation, however they share some features, e.g. for both the central region ($z \approx 0$) is disfavored; also the areas with small ξ and large $|\cos\theta|$ are less favorable in both graphs. This leads to the following interpretation of the different regions of the phase space. The region where only the quasiparticles of gap 2Δ are present corresponds *grosso modo* to the blocking region of the exact dispersion law, whereas the region where both species of quasiparticles (i.e. with gaps Δ and 2Δ) are present corresponds to the pairing region of the exact dispersion law. This interpretation is supported by the numerical results for the gap equation. In fact we find that the contribution of the former region (corresponding to blocking) to the gap equation is only 20% of the total.
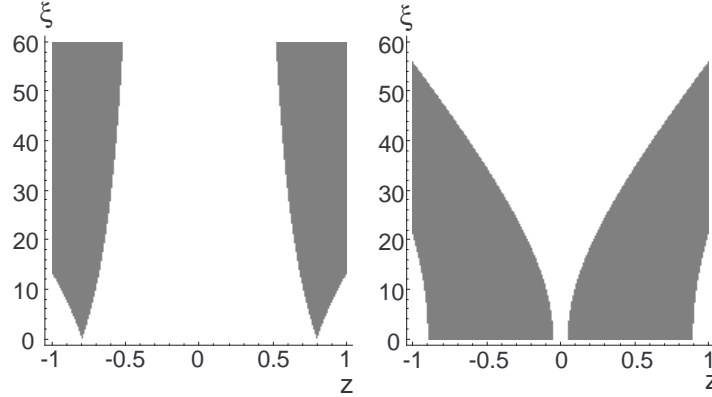

FIG. 3: On the left: Pairing (in gray) and blocking (in white) regions as computed by the exact dispersion law (50) for the strip. On the right: The gray region depicts the area where coupling is more favourable (here both quasiparticles with gap Δ and 2Δ contribute); in the white area only quasiparticles with gap 2Δ can be present. On the horizontal axis $z = \cos\theta$; on the vertical axis the longitudinal momentum measured from the Fermi surface, ξ (in MeV). Here $\Delta = 0.75\Delta_0$ $z_q = 0.8$, $\delta\mu = \delta\mu_1$.

B. Octahedron and Face Centered Cube

For the octahedron (i.e. body-centered-cube, bcc) we have $P = 6$ in (35) and we take

$$\begin{aligned} \mathbf{n_1} &= (+1,0,0), \quad \mathbf{n_2} = (0,+1,0), \quad \mathbf{n_3} = (0,0,+1), \\ \mathbf{n_4} &= (-1,0,0), \quad \mathbf{n_5} = (0,-1,0), \quad \mathbf{n_6} = (0,0,-1) \end{aligned} \tag{52}$$

in (36). For the face centered cube (fcc) we have $P = 8$ and

$$\begin{aligned} \mathbf{n_1} &= \frac{1}{\sqrt{3}}(+1,+1,+1), \quad \mathbf{n_2} = \frac{1}{\sqrt{3}}(+1,-1,+1), \\ \mathbf{n_3} &= \frac{1}{\sqrt{3}}(-1,-1,+1), \quad \mathbf{n_4} = \frac{1}{\sqrt{3}}(-1,+1,+1), \end{aligned}$$

$$\mathbf{n_5} = \frac{1}{\sqrt{3}}(+1+,1,-1), \quad \mathbf{n_6} = \frac{1}{\sqrt{3}}(+1,-1,-1),$$

$$\mathbf{n_7} = \frac{1}{\sqrt{3}}(-1,-1,-1), \quad \mathbf{n_8} = \frac{1}{\sqrt{3}}(-1,+1,-1) . \tag{53}$$

The topological structure of the different regions P_k is more involved for the cubes than for the strip. We simply offer
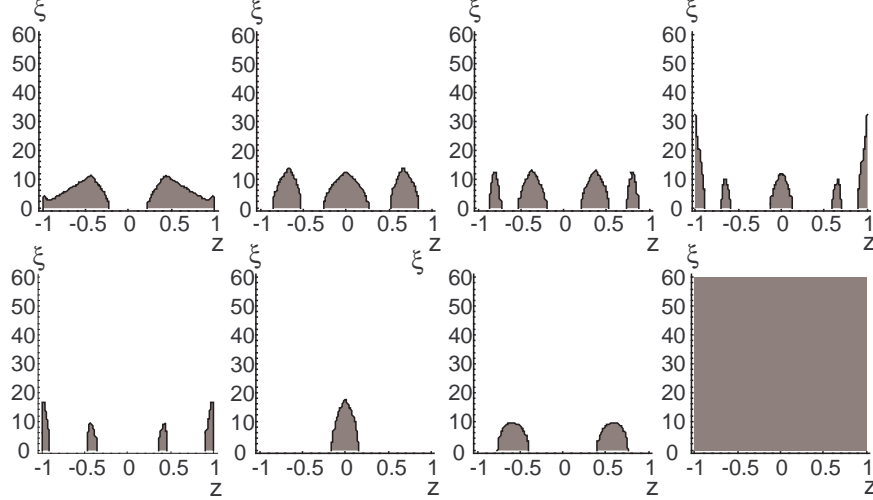

FIG. 4: The pairing regions $P_1, \ldots, P_8$ (from left to right and from top to bottom) for the face centered cube in the (z, ξ) plane for a fixed value of the polar angle $\varphi = .5$ rad. In each figure the grey area corresponds to the pairing region. Here $\delta\mu = \delta\mu_1$, $\Delta = 0.21\Delta_0$, $z_q = 0.9$. Note that for this particular value of the parameters the P_8 region covers all the Fermi surface.

in fig. 4 a picture of the different pairing regions $P_1, \ldots, P_8$ for the face centered cube in the (z, ξ) plane at fixed polar angle $\varphi = 0.5$ rad. In each figure the grey areas correspond to the pairing region.

C. Numerical results

We now present numerical results for four crystalline structures (FF, strip, bcc, fcc) that are obtained solving Eq.(38) with the appropriate set of unit vectors $\mathbf{n_m}$. In Table I we report the results obtained at the Clogston point $(\delta\mu = \delta\mu_1)$. Here P is the number of plane waves of each structure; the table shows that, among the four considered cases, the favored structure is the octahedron $(P = 6)$.

P	z_q	$\frac{\Delta}{\Delta_0}$	$\frac{2\Omega}{\rho\Delta_0^2}$
1	0.78	0.24	-1.8×10^{-3}
2	1.0	0.75	-0.08
6	0.9	0.28	-0.11
8	0.9	0.21	-0.09

TABLE I: The gap, $z_q = \delta\mu/q$ and the free energy at $\delta\mu = \delta\mu_1 = \Delta_0/\sqrt{2}$ for different crystalline structures.

We have analyzed also the case $\delta\mu \neq \delta\mu_1$. In fig. 5 we report the plot of the free energies of the octahedron (dashed line) and of the fcc (full line) phases as a function of $\delta\mu/\Delta_0$.

We find that the octahedron is the favored structure up to $\delta\mu \approx .95\Delta_0$. For larger values of $\delta\mu < 1.32\Delta_0$ the favored structure is the fcc. In Table II we report numerical results for $\delta\mu_2$ for each crystalline structure and the computed order of the phase transition between the LOFF and the normal phases. We have reported also the values of z_q and of the discontinuity in Δ/Δ_0 at $\delta\mu = \delta\mu_2 - 0^+$. The values of z_q and Δ are determined minimizing the free energy.

As remarked above, for the strip, in our approximation neither the order of the transition nor the point where the transition occurs coincide with those obtained within the Ginzburg-Landau approximation [1]. The difference in $\delta\mu_2$ is $\sim 10\%$ and in z_q is $\sim 17\%$ (the value for z_q is 0.83).
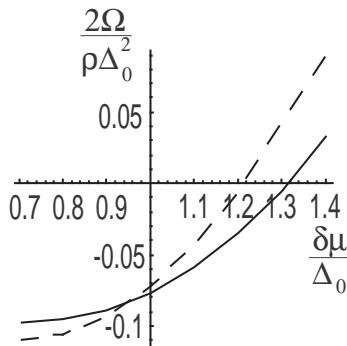
FIG. 5: The values of the free energies of the octahedron (dashed line) and of the fcc (full line) crystalline LOFF structures as a function of $\delta\mu/\Delta_0$. The octahedron is the favored structure up to $\delta\mu \approx .95\Delta_0$; for $.95\Delta_0 < \delta\mu < 1.32\Delta_0$ the fcc is favored. Here, for each value of $\delta\mu$, the values of z_q and Δ are those that minimize the free energy.

P	$\delta\mu_2/\Delta_0$	Order	z_q	Δ/Δ_0
1	0.754	II	0.83	0
2	0.83	I	1.0	0.81
6	1.22	I	0.95	0.43
8	1.32	I	0.9	0.35

TABLE II: The values of $\delta\mu_2$, $z_q = \delta\mu/q$, the discontinuity of Δ/Δ_0 and the order of the phase transition between the LOFF and the normal phases for different crystalline structures.

V. DISCUSSION AND CONCLUSIONS

In this paper we have discussed the gap equation for different Fulde-Ferrel-Larkin-Ovchinnikov phases at $T = 0$. Generally speaking this is a quite complicated problem and the only soluble case corresponds to the original Fulde-Ferrel phase with a single plane wave. Typically the discussion of the gap equation is made via the Ginzburg-Landau approximation which, however, holds only in proximity of a second order transition. However, the analysis of Ref. [14] shows that in many interesting cases there is no second order transition to the normal phase. Therefore, in this paper, we have developed a new kind of approximation to overcome the problems of the Ginzburg-Landau treatment. In fact, our approximation holds in the opposite case, that is when the gap is not too small. The main feature of our scheme is a convenient average over the sites defined by the crystalline structure of the condensate. The description one obtains amounts to the definition of a multi-valued gap function having P branches. Each of the branches has a gap given by $k\Delta$, $k = 1, \cdots, P$, with Δ the solution of the gap equation. Furthermore each non vanishing value of the gap defines a particular region in the phase space, the pairing region P_k. It is interesting to notice that each region P_k depends on the actual value of Δ and therefore its very definition depends on the solution of the gap equation itself.

We have noticed that the regions P_k do not represent a partition of the phase space since it is possible to have at the same point quasi-particles with different gaps. This observation would allow to formulate the problem in terms of an effective Lagrangian containing P velocity-dependent fields $\psi^{(k)}$, each of them defined in the region P_k, and therefore with gap $k\Delta$. We will come back to this effective Lagrangian in future work.

The gap equation has been solved numerically. We have first analyzed the case $P = 1$ where our approximation is exact, and then the cases $P = 2$ (two antipodal plane waves, or strip), $P = 6$ (the octahedron or body-centered-cube) and $P = 8$ (the face-centered-cube). Our main result is that, at $T = 0$, and up to $\delta\mu \approx 0.95\Delta_0$ the energetically favored crystalline structure is the octahedron ($P = 6$). For larger values of $\delta\mu/\Delta_0$ the fcc becomes the most favored structure up to the transition to the normal state occurring at $\delta\mu_2 = 1.32\Delta_0$.

Also, in all the cases $P = 2, 6, 8$, we have found that the transition to the normal state is a first order one.

It is difficult to compare our results to previous work since we have studied a region of the phase diagram not accessible to the Ginzburg-Landau approximation. In any event let us summmarize some of the previous results. Within the GL approximation, the authors of Ref. [1] found that the strip was favored in comparison to the one-plane-wave. Also in Ref. [14] a GL analysis is performed. These authors study a much larger number of crystalline structures and conjecture that the most stable case is the face-centered cube; however no second order transition is

observed for this structure and this makes questionable the Ginzburg-Landau approximation. As for the octahedron, the phase transition between the LOFF and the normal phase is also first-order. These results on the order of the transition for $P = 6, 8$ agree with our approximation. Finally, in a recent paper [21], the analogous problem has been studied in the context of the quasiclassical equations for superconductors by using a Fourier expansion. These authors get that the favored state at zero temperature is the octahedron, $P = 6$, and also find that the transition to the normal state is first order. Therefore the results of their analysis agree with our approximation, at least qualitatively.

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